Integration Theory
The Henstock–Kurzweil Integral:
Its Essence, and the Generalization of Riemann

Professor George Huamán Baca

Although the Riemann integral expands the class of integrable functions, it possesses well-known limitations. One of its requirements is that the function to be integrated must be bounded and defined on a closed, bounded interval. There exist derivatives that are not integrable by Riemann's method; that is, within Riemann's framework, not every derivative function possesses a primitive—among other shortcomings. The Riemann–Stieltjes integral corrects some of these deficiencies, and Lebesgue's integral corrects most of them. By contrast, the Henstock–Kurzweil integral not only integrates all Riemann-integrable functions, but also integrates broader classes of functions. Indeed, there exist bounded functions that remain non-integrable in Riemann's sense.

It is known that the Riemann integral integrates every continuous function, as well as every discontinuous function whose set of discontinuity points has measure zero.

The French mathematician Henri Léon Lebesgue shifted the partition from the domain to the range of values and, through a construction based on the measure theory he himself developed, defined his integral, which turns out to be far more powerful than Riemann's, generalizing it substantially. Even so, Lebesgue's integral retains certain limitations, which to a significant extent are resolved by the Henstock–Kurzweil integral.

This new integration process employs a positive function called a gauge function—a subtle but profound modification of Riemann's definition, replacing the constant δ with a point-dependent $\delta(x)$. Here, δ is no longer a constant, but a strictly positive function that adapts to the local behavior of the integrand. Through this change, Henstock and Kurzweil achieve a genuine generalization of the Riemann integral. Moreover, they resolve the long-standing problem of the Fundamental Theorem of Calculus: now, every derivative is integrable, and the integral recovers the original function up to a constant.

The core insight provided by Cousin's lemma—which is equivalent to the supremum axiom and the Archimedean property—and therefore the very essence of the HK integral, is this: it takes a convenient subinterval, or indeed the entire partition from Riemann's definition, and breaks it apart once more until we obtain a subinterval, or subintervals, whose size is smaller than an arbitrarily small number—what we might

call “the epsilon of an ε, or the delta of a δ.” That is, we arrive at what in Quechua-influenced mathematical Spanish we may term a *subintervalito*: an interval already small, yet iteratively refined into something practically infinitesimal—akin, in spirit, to the fine-scaled structure of the Cantor (fractal) set.

These *subintervalitos*, of effectively negligible length, are placed as tight neighborhoods around points where the function f to be integrated is not continuous or is unbounded—precisely those points where the Riemann integral encounters obstruction.

Thus, if at some point a in the domain of f the function is unbounded or discontinuous, we apply Cousin’s lemma to the subinterval containing a, partitioning it further, and then we ‘surround’ this point a with that *subintervalito* (of δ-fine size) whose total measure we can make as small as desired. The point a becomes an accumulation point and the center of a conceptual “ball” (the *subintervalito*). We tag the point a itself—or, in cases where f is unbounded at a, we tag nearby points where $|f|$ grows large without becoming infinite.

Now, because the base of the extremely thin rectangle formed around the problematic point a has length (measure) that tends to zero, while its height is (a) or $f(a \pm \delta)$, the product “base × height” tends to zero. In essence, the ball centered at a tightens progressively until it finally ‘stifles’ the problematic point.

The ball—or *bolita*—has a δ-fine radius. As can be seen, δ acts as a stifling function for elements (points) that cause difficulty.

Where the Riemann integral meets obstruction, we proceed exactly as described. And in this way, integration is accomplished under the HK method.

If we were to apply the notion of a fuzzy set to the subinterval containing the problematic point, might the explanation become even clearer?

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